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Factor Investing and forex Portfolio Management

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Abstract

Lustig, Roussanov, and Verdelhan (2011) have recently introduced the "dollar risk factor"(DOL) and the "carry trade factor"(HML), and show that they can price carry trade portfolios, in the cross-section. This new result is useful not just in the academic literature on cross-sectional asset pricing, but also in risk management and portfolio optimization, as the same factors are widely used in the industry.

In this paper, we test the relevance of these factors in contributing to a diversified forex portfolio and risk management. It is surprising that very little has been done on this important issue. We shall try to fill this gap. In contrast to the existing literature we first consider a large and detailed study to investigate the effect of introducing asymmetry and time-varying effects amongst the factors, thereafter we measure their economic adds value to a forex portfolio in terms of fx investment allocation and risk management.We show that modelling non-linear dependency is important and adds value to a forex portfolio.

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1 Introduction

Lustig, Roussanov, and Verdelhan (2011) have recently proposed two forex factors, the "dollar risk factor"(DOL) and the "carry trade factor"(HML) to explain the returns of a carry trade portfolio as compensation for risk. They form two empirically motivated factors, the DOL factor which is the cross-sectional average of currency excess return, and the HML factor which is the return of high-yield currencies minus the return of low-yield currencies and they show that these factors can price carry trade portfolios in the cross-section. Menkhoff et al. (2012) investigate the profitability of momentum strategies and find that the excess return of these strategies, in the cross-section, is impressive. Kroencke, Schindler, and Schrimpf (2014) use forex VAL (Value factor), momentum and carry trade and show that investing in these factors can enhance the risk-return profile of an international portfolio. The forex factors discussed above have become so pervasive in the literature that practitioners (hedge funds) have now started to consider them for forex portfolio risk management.

In this paper, in constrast to the papers cited above, we do not aim to explain the excess returns (in the cross-section) of carry trade or momentum strategies. We focus on currencies portfolios and model the dependence structure amongst the forex factors. We study the economic benefit (cost), in terms of portfolio performance and risk, for a hedge fund in using these factors for managing an forex portfolio and assuming different scenarios for the portfolio's dependence structure. We show strong empirical evidence of asymmetric and time-varying correlation across the factors [1]. We show that this result has important consequences for portfolio management, and risk management as it implies that linear correlation overstates the benefits of diversification. We stress that, unlike most of the literature in forex which focuses on the determinants of expected currency returns, our contribution rests on investigating the factor co-movement and its economic value to a forex portfolio. While the popularity of the forex factors cited above has grown exponentially, not just in the academic literature but also amongst practitioners, very little has been done to study them from the perspective of forex asset allocation or risk management. We shall try to fill this gap.

What we do in this paper has been largely discussed in the equity field. For example, there is extensive literature on how well the typical Fama and French (1993) pricing factors capture expected stock returns (see for example Fama and French (1993)'s market, size and value pricing factors and others). While most of these papers assume orthogonality across

[1]While we aknowledge that understanding the economic drivers of the dependence structure amongst the factors is an interesting topic and indeed we have it on the agenda for future research, we do not address it in this paper as we mainly focus here on portfolio analysis and risk management.

pricing factors,[2] Christoffersen and Langlois (2013) use a different approach and model the co-movement across the three Fama and French (1993) factors using non-linear models. They show that asymmetry and time-varying dependence across the factors is important and that the utility gain for a risk-averse investor increases significantly. In contrast, very little has been done in the forex market. In the first part of this paper, we consider a very large and detailed analysis on the dependence structure of the forex factors and show that some of the most widely used forex factors, VAL, DOL and HML for example, are not orthogonal even in terms of linear dependence, and this has a crucial implication for forex portfolio management. In the second part of the paper, we build on these results and show that a well diversified currency portfolio should indeed take into account non-normality in returns. To do this, we extend the dynamic copula models used in the equity market in Christoffersen and Langlois (2013) to model the non-linear dynamic dependence structure of forex factors. We find supportive empirical evidence in favour of the time-varying non-linear dependence amongst the forex factors and we show that such a dependence structure can be properly modelled by the dynamic skewed t copula model of Christoffersen and Langlois (2013).

An additional contribution of this paper is in assessing the economic value of modelling the dependence amongst the forex factors in the context of portfolio management. To do so, we discuss two examples: a portfolio approach and estimate optimal weights for a risk-averse investor taking a position in the forex factors; we extend the model of Patton, Ziegel, and R. Chen (2019) to capture joint dependence across the factors and forecast the VaR and Expected short-fall of an forex portfolio. The linear correlation for the multivariate normal distribution is used as a benchmark. We find significant utility gains when using non-linearities. We believe that our large and detailed study on the dependence structure of forex factors combined with the forex portfolio analysis and risk management are novel contributions and the empirical results discussed are very relevant not only in academia but also to central banks and hedge funds.

Our paper is similar in spirit to Barroso and Santa-Clara (2015b) although its main contribution is on carry trade and momentum in optimizing a forex portfolio. They find that carry, momentum and reversal all contribute to optimized portfolio performance. We show that this is also the case for the factors considered in this paper, not only in enhancing portfolio performance but also managing risk. Additionally to this, Barroso and Santa-Clara (2015b) do not model the dependence amongst the factors. We show that asymmetry and the time-varying dynamics amongst the forex factors is key, and neglecting it has a significant economic cost.

[2] This is a well documented empirical fact induced by the linear correlation coefficients.

Jordà and Taylor (2012) show that the 2008 financial crisis impacted negatively on different strategies, including the carry trade strategy. They combine carry trade momentum and the real exchange rate and show that this strategy can generate positive in-sample and out-of-sample returns. Our paper is also related to Jordà and Taylor (2012), but we study extensively the dependence structure across the factors and the way this affects the fx asset allocation and portfolio risk management.

Finally, our paper is also related to the extensive literature on the non-linear correlation in forex. For example, Scotti and Benediktsdottir (2009) report strong evidence that foreign and US recessions are important for explaining the joint dependence structure across the tendency of currencies with higher interest rate differentials to move less closely together, not only on average (correlation), but also when extreme events occur (tails). Patton (2006) finds evidence that the mark-dollar and yen-dollar exchange rates are more correlated when they are depreciating against the dollar than when they are appreciating. This asymmetry could be induced by the asymmetric responses of central banks to exchange rate movements. We show strong evidence of asymmetry and time-varying correlation across some recently proposed forex factors and estimate the cost to portfolio management of ignoring it. To achieve this goal, we rely on a parsimonious copula model which allows us to model the dependence across the four factors cited above.

The paper is organized as follows: section 2 describes our data; section 3 introduces the univariate modelling and pairwise correlation analysis among forex factors; section 4 presents the joint distribution modelling of forex factors by using Copula models; section 5 introduces the economic implication of the copula model by constructing optimal portfolios for risk-averse investors; section 7 provides the conclusion.

2 Data

We use weekly forward and spot rates from January 1, 1989, to March 20, 2020, for 31 active trading currencies. The data are all from DATASTEAM. The excess return of carry trade is calculated using term t log forward rate less term $t+1$ log spot rate for each currency. We now discuss how the currency factors (DOL,HML, MOM and VAL) have been constructed. The DOL factor is the mean of the 31 currencies' excess return. This is what we denote as the dollar risk factor. In constructing the HML factor, we follow Lustig, Roussanov, and Verdelhan (2014) and sort the currency returns from lowest to highest based on the forward premium and allocate them into five portfolios. The HML factor is the difference between

[Figure 1 of time series plot of factor values is about here]

the mean returns of the fifth portfolio (the largest forward premium) and the first portfolio (the smallest forward premium). We denote it as the carry trade factor. For the momentum (MOM) factor, we follow Menkhoff et al. (2012) and use the previous 6-week formation period and 1-week holding period to sort the currencies into five portfolios based on their lagged returns. The MOM factor is the difference between the mean returns of the lowest lagged return portfolio and the highest lagged return portfolio. Finally, we compute the VAL factor following Kroencke, Schindler, and Schrimpf (2014)

$$Q_{j,t} = \frac{S_{j,t}P_{j,t}}{P^*_{j,t}} \tag{1}$$

where $P_{j,t}$ denotes the price level of consumer goods in country j at term t, and $P^*_{j,t}$ the corresponding foreign price level (here is USD). While the $Q_{j,t}$ denotes the real exchange rate of country j at term t.

$$F_{VAL,j,t} = \left(\frac{Q_{j,t-3}}{Q_{j,t-13}} - 1\right)(-1) \tag{2}$$

The VAL factor can be calculated using the average real exchange rate over 3 and 13 weeks. We then sort the currency returns from lowest to highest based on the VAL factor and allocate them into five portfolios to obtain the VAL portfolio.

3 Currency Market Factors

3.1 Descriptive statistics

In this section, we present some descriptive statistics for the 4 portfolios (DOL, HML, MOM and VAL). Figure 1 graphs the time series plot of the factors. There is the clear presence of a volatility cluster during the 2008 financial crisis period and this seems to be more evident for MOM and VAL factors.

We report the descriptive statistics in Table 1. We include the annualized sample mean, the Newey-West standard error adjusted test statistics, the annualized standard deviation, the

[Table 1 factor descriptive stats table is about here]

[Figure 2QQ plot is about here]

skewness, kurtosis, autocorrelation coefficient and linear correlation matrix. The annualized mean return is the highest for the carry trade factor HML and is negative for the DOL factor. It is interesting that for the full dataset all factors show excess kurtosis and the skewness is negative for all factors but positive for VAL. The second panel shows the autocorrelation coefficients. Most of the factors, apart from the DOL, have strong second-order and third-order autocorrelation. This autocorrelation could cause the non-normality shape observed in Figure 1.

We report the sample linear correlation matrix in the last panel. There are significant pairs of correlations among all factors. We observe a negative correlation between MOM and DOL. This result is also documented in equity momentum studies, (see for example Daniel and Moskowitz (2016)). We surprisingly find that correlation between HML and MOM is positive and HML and VAL is negative. Since most studies have reported a negative correlation between HML and MOM factors, we investigate this issue further and split the full sample into two: one part including the 2008 finnacial crisis and one not including it. Table 2 shows the results. The financial crisis does not seem to be causing that result, (see Table 1). However, when we split the sample into developed and developing countries, we find a clear difference for correlations of HML and MOM or HML and VAL. In developed countries, the factors HML and MOM have the expected negative correlation while the HML and VAL are positively correlated.

Further evidence of non-normality can be seen in Table 1, while Figure 2 complements and supports that evidence. In fact, factors' empirical quantiles diverge significantly from a normal distribution.

The empirical evidence above, although in a simple form, seems to support our main idea: correlation amongst forex factors is not captured by a normal distribution. In the next sections we shall investigate this issue in more detail and investigate its implications.

3.2 Modelling Dependence Amongst the forex Factors

In this section we conduct a more detailed analysis of the dependence amongst the forex factors. We model the dependence structure for each pair of currency factors using threshold correlations or quantile dependence, as in Christoffersen and Langlois (2013). [3] The idea here is to characterise the dependence of two variables in the joint lower or joint upper tails, respectively. Unlike linear correlation, this approach involves modelling the asymmetric dependence structure between extreme events, which is appropriate in the presence of skew and excess kurtosis observed in Table 1. We define the threshold correlation $\overline{\rho}_{i,j}(u)$ for any two factors i and j as follows:

$$\overline{\rho}_{i,j}(u) = \begin{cases} corr(r_i, r_j | r_i < F_i^{-1}(u), r_i < F_i^{-1}(u)) & when\, u \leq 0.5 \\ corr(r_i, r_j | r_i \geq F_i^{-1}(u), r_i \geq F_i^{-1}(u)) & when\, u \geq 0.5 \end{cases} \tag{3}$$

Where u is a threshold between 0 and 1, and $F_i^{-1}(u)$ is the empirical quantile function of the univariate distribution of r_i.

Figure 3 shows, on the left, the scatter plot of two factors. Alongside we plot the empirical threshold correlation against the threshold u for the same pair of factors.[4] As a comparison, we assume that the theoretical threshold correlation, given the factors pairs, follows a bivariate normal distribution (see the dashed line). For bivariate normal distributions, the threshold correlation will be symmetric around 0.5 and will gradually approach 0. Figure 3 shows that the bivariate normal assumption does not hold, as we observe increasing correlations in extreme events. The empirical correlations show a significant degree of asymmetry, especially in the tail. Correlations amongst factors appear to be, in general, positive and large [5]. It is worth discussing, for example, the conditional correlation between the carry and momentum factors in Figure 3. In fact, in the literature, it is standard that these two factors are independent when an unconditional measure of correlation is used. Figure 3 shows that these two factors might be very highly (positively) correlated instead. Consider now the factors DOL and HML. Again, the (conditional) correlations are large and most of the times positive. This result might be relevant for most of recent fx (cross sectional) asset pricing studies that assume independence between the Dollar risk and carry risk factors.

In sum, our results show that assuming linear dependency amongst the factors will lead to underestimating portfolio risk, in extreme event scenarios, and so diversification, in this

[3] The same method was used by Longin and Solnik (2001), Ang and J. Chen (2002), Ang and Bekaert (2002) and Patton (2004)

[4] We follow Christoffersen and Langlois (2013) who compute the threshold correlation when at least 20 pairs of values are available.

[5] Although these results appear rather interesting and worthy of further investigation, this is not the objective of this paper and we leave this question for future research

[Figure 3 Threshold correlation graph about here]

[Figure 4 Autocorrelation graph about here]

case, will not work in reducing the overall risk exposure. The empirical results above are important and new as they shed new light on the literature (see for example Kroencke et al, 2014; Brandt et al , 2009) and show that dependence amongst the forex fatcors is very significant.

3.3 Univariate Modelling

The empirical results in Table 1 also show that autocorrelation could be an important issue for factors' returns. In Figure 4, the autocorrelation function is plotted by a dashed line for all the factors up to 100 lags, a 95% confidence boundary included. Financial time series are generally subject to heteroscedasticity and volatility clustering. We plot the autocorrelation function for the absolute value of the factors on the same graph. We find a strong and persistent serial correlation.

We model the dynamics of our factors by using a univariate autoregressive-non-linear generalized autoregressive conditional heteroscedasiticity (AR-NGARCH) process. The conditional mean is estimated by an AR(1) process as follows:

$$r_{j,t} = \phi_{0,j} + \phi_{i,j} r_{j,t-1} + \sigma_{j,t}\epsilon_{j,t} \tag{4}$$

Where $r_{j,t}$is the factor value of factor j at time t. The conditional volatility is governed by an NGARCH (R. F. Engle and Ng 1993)

$$\sigma^2_{j,t} = \omega_j + \beta_j \sigma^2_{j,t-1} + \alpha_j \sigma^2_{j,t-1}(\epsilon_{j,t-1} - \theta_j)^2 \tag{5}$$

The NGARCH model allows for the asymmetric influence of past return innovations $\epsilon_{j,t-1}$. Since financial time series generally show a "leverage effect", an unexpected drop in return may have a bigger impact on conditional volatility than an unexpected increase (i.e. $\theta_j$is positive). Under this circumstance, the NGARCH model is expected to mitigate the skewness and excess kurtosis. We use the maximum likelihood method under the assumption of i.i.d. normal innovations of $\epsilon_{j,t}$.

[Table 3 Estimation table of normal residuals about here]

[Table 4 Estimation table of skewed t residuals about here]

Table 3 reports the coefficient estimates and diagnostic tests under the normal assumption for $\epsilon_{j,t}$. In the first panel, we report the estimated coefficients and standard errors of an AR(1)-NGARCH model ϕ_0,ϕ_1,α,β and θ. The parameters (ϕ_0) are all significant except for the DOL. Most parameters of the NGARCH model are also significant. The coefficient θ of the VAL and the MOM factors have large positive values which are statistically significant while the DOL factors have insignificant negative θ. The log-likelihoods are all significant and positive.

The divergence between model skewness/kurtosis points towards strong non-normality of ϵ_j.To better model the factor dynamics, we employ the skewed t distribution of Hansen (1994) for error term $\epsilon_{j,t}$, where the coefficients $\kappa_j$and $\upsilon_j$govern the skewness and the kurtosis. We use the maximum likelihood method under the assumption of skewed t distribution of $\epsilon_{j,t}$to estimate the AR(1)-NGARCH model. The results are reported in Table 4 which shows that the kurtosis parameters (υ) are all significant and the skewness factors (κ) of HML are not significant.[6]

Figure 5 graphs the autocorrelation function for the residual and its absolute value. After adjusting the skewness and excess kurtosis by assuming a normal distribution, the serial correlation in absolute value is highly reduced. Figure 6 is the QQ plot of the residuals from skewed t AR(1)-NGARCH. When comparing these results with Figure 2, we see that most of the skewedness and kurtosis have been modelled after using the AR(1)-NGARCH.

4 Modelling Asymmetry Amongst the Forex Factors

The empirical evidence above supports the presence of non-normality and asymmetry in the threshold correlation. To account for these features, we use copula models as in Patton (2006). We use this methodology as it is a flexible framework to characterise multivariate distributions. The joint probability density function $f_t(r_{1,t+1}, ..., r_{N,t+1})$ of the N forex pricing factors can be decomposed as follow:

[6] By comparing the significance for the whole AR(1)-NGARCH model in Table 4, we find that the AR(1)-NGARCH model with the normal distribution fits the data well.

[Autocorrelation graph of residual series about here]

[QQ plot of residuals about here]

$$f_t(r_{1,t+1},...,r_{N,t+1}) = c_t(\eta_{1,t+1},...,\eta_{N,t+1}) \prod_{j=1}^{N} f_{j,t}(r_{j,t+1}), \tag{6}$$

Where $f_{j,t}(r_{j,t+1})$ is the univariate marginal probability density function for factor j and time t; $c_t(\eta_{1,t+1},...,\eta_{N,t+1})$ is the conditional density copula function;$\eta_{j,t+1}$ is the marginal probability density for factor j.

$$\eta_{j,t+1} = F_{j,t}(r_{j,t+1}) \equiv \int_{-\infty}^{r_{j,t+1}} f_{j,t}(r)dr \tag{7}$$

We follow the univariate skewed t AR(1)-NGARCH model given in Section 3.3. The $F_{j,t}$ is the cumulative distribution function (CDF) of the skewed t distribution of Hansen (1994).

4.1 Copula Models

Patton (2006) discusses the flexibility of copula models and shows that this methodology can capture observed empirical facts in the forex market, for example correlation structure for currencies against the US Dollar is stronger when the currency depreciates than when it appreciates. Therefore, in our case, copula models help us to estimate the joint dynamic distribution of the factors.

We shall introduce the copula model in this section. The most common functional forms of copula models in financial time series are normal copula and student t copula. However, these two copula models can only generate symmetric multivariate distributions and fail to account for the asymmetry in threshold correlations that we have empirically shown above for the factors. Copulas from the Archimedean family (The Clayton, the Gumbel and Joe-Clayton specifications) can be used for asymmetric bivariate distributions, but they are not easily generalized to high dimensional cases.

Demarta and McNeil (2005) propose the skewed t distribution and the skewed t copula which have been widely used in financial modelling.[7] The skewed t distribution belongs to

[7] The skewed t copula is used by Christoffersen, Errunza, et al. (2012) for the analysis of international equity diversification and Christoffersen and Langlois (2013) for equity market factor modelling. Cerrato et al. (2017b) use this model for joint credit risk analysis of UK banks. Cerrato et al. (2017a) model the higher-order components of equity portfolios.

the multivariate normal variance mixtures class. An N-dimensional skewed t random variable X has the following representation:

$$X = \sqrt{W}Z + \lambda W \tag{8}$$

Where W follows an inverse Gamma $IG(\upsilon/2, \upsilon/2)$ distribution; Z is a N-dimensional normal distribution with mean 0 and correlation matrix Ψ; λ is a $N \times 1$ asymmetry parameter vector. The multivariate probability density function of the skewed t distribution is:

$$f_t(r; v, \lambda, \Psi) = \frac{2^{\frac{2-(v+N)}{2}} K_{\frac{v+N}{2}}\left(\sqrt{(v + z^{*\top}\Psi^{-1}z^*)\lambda^\top\Psi^{-1}\lambda}\right) e^{z^{*\top}\Psi^{-1}\lambda}}{\Gamma\left(\frac{v}{2}\right)(\pi v)^{\frac{N}{2}} \mid \Psi \mid^{\frac{1}{2}} \left(\sqrt{(v + z^{*\top}\Psi^{-1}z^*)\lambda^\top\Psi^{-1}\lambda}\right)^{-\frac{v+N}{2}} \left(1 + \frac{z^{*\top}\Psi^{-1}z^*}{v}\right)^{\frac{v+N}{2}}} \tag{9}$$

The copula density function derived from the above probability density function is:

$$\begin{aligned} c_t(\eta; \lambda, v, \Psi) \quad &= \quad \frac{2^{\frac{(v-2)(N-1)}{2}} K_{\frac{v+N}{2}}\left(\sqrt{(v + z^{*\top}\Psi^{-1}z^*)\lambda^\top\Psi^{-1}\lambda}\right) e^{z^{*\top}\Psi^{-1}\lambda}}{\Gamma\left(\frac{v}{2}\right)^{1-N} \mid \Psi \mid^{\frac{1}{2}} \left(\sqrt{(v + z^{*\top}\Psi^{-1}z^*)\lambda^\top\Psi^{-1}\lambda}\right)^{-\frac{v+N}{2}} \left(1 + \frac{z^{*\top}\Psi^{-1}z^*}{v}\right)^{\frac{v+N}{2}}} \\ &\times \prod_{j=1}^{N} \frac{\left(\sqrt{\left(v + \left(z_j^*\right)^2\right)\lambda_j^2}\right)^{-\frac{v+1}{2}} \left(1 + \frac{\left(z_j^*\right)^2}{v}\right)^{\frac{v+1}{2}}}{K_{\frac{v+1}{2}}\left(\sqrt{\left(v + \left(z_j^*\right)^2\right)\lambda_j^2}\right) e^{z_j^*\lambda_j}} \end{aligned} \tag{10}$$

Where K$(\cdot)$ denotes the modified Bessel function of the second kind, and $z^* = t_{\lambda,v}^{-1}(\eta_i)$ denotes the copula shocks where $t_{\lambda,v}(\eta_i)$ is the univariate skewed t distribution:

$$t_{\lambda,v}(\eta_i) = \int_{-\infty}^{\eta_i} \frac{2^{1-\frac{v+1}{2}} K_{\frac{v+1}{2}}\left(\sqrt{(v + x^2)\lambda_i^2}\right) e^{x\lambda_i}}{\Gamma\left(\frac{v}{2}\right)\sqrt{\pi v}\left(\sqrt{(v + x^2)\lambda_i^2}\right)^{-\frac{v+1}{2}} \left(1 + \frac{x^2}{v}\right)^{\frac{v+1}{2}}} dx \tag{11}$$

However, a closed-form solution for skewed t quantile function is not available. We use simulation to define the quantile function and employ 1,000,000 replications of equation 8.

4.2 Modelling Dynamic Dependence Amongst the forex Factors

Another interesting feature of the results above is that correlations change over time. We now discuss how we account for this feature. The difference between dynamic model and constant model whether the correlation of factors are constant or not. Following Christoffersen, Errunza, et al. (2012) and Christoffersen and Langlois (2013), we use R. Engle (2002)'s dynamic conditional correlation (DCC), where the correlation matrix dynamic is generated as 12

$$Q_t = Q(1-\beta_c-\alpha_c)+\beta_c Q_{t-1}+\alpha_c z_{t-1} z_{t-t}^T \tag{12}$$

In the case of N pricing factors, $Q_t$is a $N \times N$ positive semi-definite matrix for time t;$\alpha_c$and β_c are scalars; $z_t$is a $N \times 1$ row vector of standardized residuals with jth entry $z_{j,t} = F_c^{-1}(\eta_{j,t})$, where F_c^{-1} is the inverse CDF from copula estimation; Q is a constant matrix which is a full-sample correlation matrix. The dynamic conditional correlation between factor i and j for time t is defined as

$$\Psi_{ij,t} = \frac{Q_{ij,t}}{\sqrt{Q_{ii,t} Q_{jj,t}}} \tag{13}$$

Coefficient $\beta_c$and α_c are estimated to allow the dynamic correlation. Note that the dynamic copula mean-reverts to the full sample correlation matrix Q. The estimates of coefficient $\beta_c$and α_c are showed in Table 5.

4.3 Estimation Method

We use a composite log-likelihood estimation inspired by R. Engle, Shephard, and Sheppard (2009) and Christoffersen, Errunza, et al. (2012).[8] The composite likelihood function in our case is defined as :

$$CL(\theta) = \sum_{t=1}^{T}\sum_{i=1}^{N}\sum_{j>i} \ln c_t\left(\eta_{i,t}, \eta_{j,t}; \theta_{i,j}\right) \tag{14}$$

[8] R. Engle, Shephard, and Sheppard (2009) find that in the large-scale DCC model, the traditional likelihood method yields biased estimates.

[Table 5 Copula results about here]

Where $\theta$is the parameter set; $c_t(\eta_{i,t}, \eta_{j,t}; \theta_{i,j})$ is the bivariate copula distribution of factor pair i and j. We maximize the composite loglikelihood function $CL(\theta)$ to get the Copula coefficient estimates $\theta_{i,j}$ for each factor pair. We then average $\theta_{i,j}$to obtain an estimator of the parameter set θ. The standard errors are based on R. Engle, Shephard, and Sheppard (2009). Following Christoffersen, Errunza, et al. (2012), all the copula models are estimated by this method. We also report the parameter estimates from maximizing the conventional likelihood function along with parameter standard error based on X. Chen and Fan (2006) in the Appendix.

4.4 Empirical Results

The first panel of Table 5 shows the composite likelihood estimates for static/dynamic parameters of normal, student t and skewed t copula. The degree of freedom ν and most of skewedness parameter λ in skewed t copular are all significant. This is consistent with the non-normal and asymmetric dependence of currency factors. For the static copula models, the full sample correlation estimates are reported. For dynamic copula models, we report DCC parameter estimates $\alpha_c, \beta_c$and long-term mean-reverting correlation matrix Q as in equation 12. The estimates of Q are about the same as for the full sample correlation of the static copula models. DCC parameters $\alpha_c$and $\beta_c$are significant in all three models. This result supports the time-varying correlation.

In the lower panel of Table 5, we report the model diagnostic statistics. We report the log-likelihood and the PLR test statistic test. The dynamic copula models display the best fit. This is consistent with the presence of time-varying correlation and asymmetric dependence. Following X. Chen and Fan (2006), we perform the pseudo-likelihood ratio (PLR) test to show that the skewed t copula model outperforms the student t copula. The null hypothesis is that the asymmetry parameters (λ) in the skewed t copula are all zero. The pseudo-likelihood ratio (PLR) test cannot reject the null hypothesis. Thus, the skewed t copula models are significantly asymmetric and different from the student t copula.

Figure 7 shows the dynamic correlation implied by the skewed t dynamic copula during the period from January 1 1989, to March 20 2020. We consider the most difficult period of the recent financial crisis. The correlations of pairs HML&VAL and HML&MOM move around the value of Q (in equation 12). During 2008, all pairs of correlation fluctuate considerably. The financial crisis hugely impacted on the forex market, invalidating models.

[Figure 7 Dynamic correlations of residuals about here]

[Figure 8Threshold Correlations for Factor Residuals and Copula Models]

To reinforce our empirical results pointing towards non-normality and checking their robustness, in Figure 8 we plot the empirical threshold correlation of residuals $z^*$from the AR-NGARCH model along with the standard bivariate normal implied threshold correlation, student t copula and skewed t copula implied threshold correlations. It is evident that the empirical threshold correlations are far from a bivariate normal distribution. In what follows we use the skewed t copula to model the dependency structure of the fx factors.

5 Economic Implication

The empirical evidence above suggests that the forex factors have significant time-varying asymmetric dependence. What is the economic cost for a forex trader to ignore this dependence structure? We shall consider two examples: forex portfolio management and forex portfolio risk management. We assess the economic value of considering this type of dependence structure in a forex portfolio. As in Kroencke, Schindler, and Schrimpf (2014) we use a real time strategy. We show first that once we implement an forex optimised strategy and consider asymmetry and time-varying, in the dependence structure, the benefit in terms of utility for the investor is largely improved and second that portolio Value at Risk and Expected Short-Fall are highly reduced. We compare a battery of models accounting for different dependence structures. For portfolio analysis, we assume that at each time t, investors allocate their wealth, based on the weighting vector w_t, across the 4 currency factors to maximize their expected utility. We compare the return characteristics of alternative strategies by using different dependence structure models and a large real time out-of-sample analysis.

5.1 The Investor's Optimization Problem

We assume that investors follow a constant relative risk aversion (CRRA) utility function:

$$U(\gamma) = \begin{cases} (1-\gamma)^{-1}\left(P_0\left(1+w_t^{\top}r_{t+1}\right)^{1-\gamma}\right) & if\,\gamma \neq 1 \\ log\left(P_0\left(1+w_t^{\top}r_{t+1}\right)\right) & if\,\gamma = 1 \end{cases} \quad (15)$$

Where P_0 is the initial wealth which we set at \$1 here, $r_t$is the vector 4 currency factor returns at time t, w_t is the weighting vector, γ denotes the degree of relative risk aversion (RRA). We consider 3 levels of RRA: $\gamma = 3, 7, 10$. The weighting vector for each time t is obtained by maximizing the expected utility function given different assumptions for the factors' joint distribution.

$$
\begin{aligned}
w_t^* &\equiv \underset{w \in W}{arg\,max}\, E_{\hat{f}_{t+1}}\left(U\left(1 + w_t^\top r_{t+1}\right)\right) \\
&= \underset{w \in W}{arg\,max} \int \frac{\left(1 + w_t^\top r_{t+1}\right)^{1-\gamma}}{1-\gamma} f_{t+1}\left(r_{t+1}\right) dr_{t+1}
\end{aligned} \tag{16}
$$

Where $f_{t+1}\left(r_{t+1}\right)$ denotes the joint distribution of 4 factors. We assume that investors face investment constraints in that the risk exposure to any single factor and the four factors in total is less than \$1. Thus the weighting matrix $w = \left\{(w_1, w_2, w_3, w_4) \in [-1, 1]^4 : \mid w_1 \mid + \mid w_2 \mid + \mid w_3 \mid + \mid w_4 \mid \leq\right.$ Due to the complexity of the joint distribution$f_{t+1}\left(r_{t+1}\right)$, solution for w_t is generally not given analytically. We solved 16 by simulating 10,000 Monte Carlo replications for the four factors using a multivariate distribution $f_{t+1}\left(r_{t+1}\right)$.

5.2 Forex Portfolio

Our weekly investment strategy is implemented in two stages: the first stage consists of modelling the dependence structure or joint distribution for the expected return $f_{t+1}\left(r_{t+1}\right)$; the second stage involves the estimation of the factor weighting vector by maximizing the investors' utility function 16 given the estimated joint distribution in the first stage. To begin with, we estimate the skewed t AR-NGARCH model (equation 45) for the four factors using the previous data sample. Thereafter, we estimate the dependence structure among the four residuals from the AR-NGARCH by using copula models.[9] Each time t, the expected factor return for factor j is generated by equation 17:

$$
r_{j,t+1} = \phi_{0,j} + \phi_{1,j} r_{j,t} + \sigma_{j,t+1}\epsilon_{j,t+1} \tag{17}
$$

Where$\phi_{o,j}$ and $\phi_{1,j}$are the AR coefficients; $\sigma_{j,t+1}$ is the 1-step-ahead forecasted conditional volatility in the NGARCH model; $\epsilon_{j,t+1}$ is simulated from the joint distribution function which

[9] We also used a multivariate standard normal distribution as a benchmark for comparison with copula models.

is characterized by the copula model. Note that the parameter estimates in the AR-NGARCH and copula models are updated once a year using the whole previous data sample. For dynamic copula models, where DCC is used to model the time-varying correlation coefficient, the factor correlation is updated weekly. We start our investment from April 1, 1994, giving us an investment period of over 25 years.

In the second stage, we use the simulated 10,000 draws from $f_{t+1}\left(r_{t+1}\right)$ to value the integral in 16. Thus maximising 16 is equivalent to maximising 18

$$w_t^* \equiv \underset{w \in W}{arg\,max} n^{-1} \sum_{i=1}^{n} U^*\left(R_{t+1,i}^*\left(w\right)\right) \tag{18}$$

where

$$R_{t+1,i}\left(w\right) = 1 + w_t^{\top} r_{t+1} \tag{19}$$

$$\varepsilon = 2.2204 \times 10^{-16}$$

and

$$R_{t+1,i}^*\left(w\right) = \begin{cases} R_{t+1,i}\left(w\right) & if\; R_{t+1,i}\left(w\right) > \varepsilon \\ 2\varepsilon\left(1 - \frac{1}{1+e^{R_{t,i}(w)-\varepsilon}}\right) & if\; R_{t+1,i}\left(w\right) \leq \varepsilon \end{cases} \tag{20}$$

$$\bar{U} = n^{-1} \sum_{i=1}^{n} U\left(R_{t,i}^*\left(w_{t-1}^*\right)\right) \tag{21}$$

$$U^*\left(R_{t+1,i}\left(w\right)\right) = \frac{100}{\mid \bar{U} \mid} U\left(R_{t+1,i}\left(w\right)\right) \tag{22}$$

The cut-off 2.2204×10^{-16} was chosen as the machine epsilon. We use the function U^* instead of U directly, since the numerical maximization routine does not work well with extremely small or large values. The $\bar{U}$does not affect the ranking of alternatives, and the 100 value is the reverting mean of the $\bar{U}$.

By maximizing equation 18, we obtain the optimal weighting vector w_t for time t. Each time t, investors liquidate the previous position and rebalance their portfolios according to w_t.

The real-time investment results about here]

5.3 Performance of Different Strategies

The empirical results based on a large battery of dependence structure models are reported in Table 6. We consider three levels for the RRA, namely $\gamma = 3$ in Panel A, $\gamma = 7$ in Panel B, and $\gamma = 10$ in Panel C. We follow Christoffersen and Langlois (2013) and Patton (2004). As the value of γ increases the risk-averse level would also increase and the turnover would decrease. The portfolio mean, volatility, skewness and kurtosis of returns for the 5 different models are given in Table 6. Following Christoffersen and Langlois (2013), we use the average return of the previous two years as the expected return of the factors. This helps us to focus on the impact of higher moments on portfolio selection. We start with the full dataset (i.e. developed and developing countries).

To assess whether allowing for asymmetry and time dependence lead to better portfolio's performance by generating better trading signasl, we also report the average turnover

$$Average\, turnover\,(\%) = \frac{100}{4T} \sum_{t=1}^{T} \sum_{i=1}^{4} \mid w_{i,t} - w_{i,t-1} \mid \tag{23}$$

The estimates are all around 12%-21%, depending on the risk aversion. These values are similar within each of the panels. This shows that the improvement in realized utility across the models is not driven by the difference in trading turnover.

We compute the certainty equivalent (CE) of the average realized utility for each strategy as follows

$$CE = U^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}\frac{(1+r_{p,t})^{1-\gamma}}{1-\gamma}\right) = \left(\frac{1}{T}\sum_{t=1}^{T}(1+r_{p,t})^{1-\gamma}\right)^{\frac{1}{1-\gamma}} \tag{24}$$

where U^{-1} is the inverse of utility function and where

$$r_{p,t} = w_{t-1}^{\top} r_t \tag{25}$$

are the out-of-sample portfolio returns.

We use the multivariate standard normal model as our benchmark. For completeness we also include the performance of an equal weighted portfolio. There is clear empirical evidence that asymmetry is economically relevant (i.e. the skewed t copula out-performs the other models). Thus, by considering asymmetry one can add value to a forex portfolio. The equally weighted forex strategy produces a very different performance from the copula strategies.

To assess whether the difference between the benchmark portfolio and skew t copula portfolio is economically significant, we apply bootstrap methods under the null hypothesis that the difference is significantly different from zero. In this way, we can infer if the actual difference shown in Table 6 is economically relevant. In each case, 1000 bootstrap draws were used to calculate the $p - value$s. The copula models always out-perform the benchmark model and the equal weighted portfolio. Furthermore, the $p - values$ show that the performace of skewed t copula portfolio is robust. To better understand the results, consider an investor with a relative risk aversion of 3. This investor is gaining 0.02375% , that is 2.375bp per week if she uses the skew t copula instead of the benchmark model. Overall, there is a robust evidence that tail dependence of the forex factors is highly affected by asymmatry and time varying dynamics. These results help to shed some lights on the modeling approach used in the literature resting mainly on normality assumptions, for example, Kroencke et al (2014).

5.4 Transaction Costs

Transaction costs can significantly reduce the performance of a trading strategy. There is empirical evidence (Menkhoff et al. 2012), for example, that the performance of a momentum strategy is highly reduced after considering transaction costs. In Table 7, we consider transaction costs to check the robustness of the results presented in the previous table. To compute the cost, we follow Barroso and Santa-Clara (2015a) and write it as:

$$c_{i,t} = \frac{F^{ask}_{i,t,t+1} - F^{bid}_{i,t,t+1}}{F^{ask}_{i,t,t+1} + F^{bid}_{i,t,t+1}} \tag{26}$$

Where $c_{i,t}$ is the transaction cost of currency i at time t. $F^{ask}_{i,t,t+1}$ and $F^{bid}_{i,t,t+1}$ denote the bid and ask price of the forward exchange rate of currency i at time t. To convert currency transaction costs into factor transaction costs, we use the same method and parameters to calculate the factor transaction cost by simply changing currency excess return to the currency transaction cost.

We consider transaction costs for combined strategies and not a “stand alone” strategy as it may well be that when we consider transaction costs for a momentum strategy, the cost offsets the return for that strategy, but when it is combined with other strategies (for example carry trade) the higher profit of this combined strategy offsets the transaction costs. Clearly transaction costs are important but, overall, the main results remain unchanged.

5.5 Performance in Developed and Developing Countries

We now split the data into developing and developed countries. We do this for several reasons: first, we aim to check whether our results are driven by country-specific factors affecting the exchange rates. Second, the benefits are known of diversifying forex portfolios by including developing countries’ exchange rates. For the developed countries the p-values reject the null hypothesis only at the 10% significance level. Thus, the rejection is weaker than in the previous tables. The annualized mean return is, generally higher for the t skew copula model while annualized volatility and skewness are unchanged across the models. The large negative skew may be an indication of crash risk. As before, if we consider an investor with a relative risk aversion of 3, she would gain 0.011% , this is 1.15bp per week using the skew t copula instead of the benchmark model.

The results for developing countries also point towards an economic gain when using a skew t copula, in general, they are weaker than the ones presented for all the countries: the benefit for our investor from using a skew t copula model in this case is only 1.94bp per week. There is an economic benefit in diversifying an forex portfolio between developed and developing markets. The annualized mean return for developing countries is higher then the one for developed countries, and the annualized volatility is also higher. Overall, the CE measure for developing countries is the highest. The equal weighted strategy shows always the worst performance, see Table 8 9.

6 Risk Management

We now consider forex portfolio risk management when asymmetry and time-varying dependence structure is accounted for. It is standard in the industry to hedge an forex portfolio via forex factors, and the results of the next sections should therefore be quite informative for hedge funds managing an forex portfolio. The idea is to hedge an forex portfolio via proxies that are easier to trade. For our specific risk factors, these can easily be easly traded via investment banks with very low transaction costs. This approach is becoming more popular in the industry and it is likely to become even more so in the near future.

We compute two risk measures: Value at Risk(VaR), which is the tail quantile of the conditional distribution of the portfolio returns; We also compute the expected short (ES) fall, which is the conditional expectation of exceeding the VaR. We use both VaR and ES to measure the risk of our portfolio. In the next sections, we shall follow Patton, Ziegel, and R. Chen (2019). We consider 9 different models. The first three correspond to a univariate distribution forecast, the next three correspond to dynamic copula forecasts, the last three show the forecasting results from NAGARCH dynamic copula models. We discuss these models in the next section.

6.1 Copula VaR and ES Forecasting

We build upon the model of Patton, Ziegel, and R. Chen (2019) and use the joint distribution of factors to forecast VaR and ES. The value at risk is a conficence level α which is opposite in sign to the value of the $(1-\alpha)$ quantile:

$$VaR_{\alpha} = -F^{(-1)}\,(1-\alpha)$$

where the $F^{(-1)}$ denotes the inverse cumulative distribution function.

We apply the distribution of the asset returns, the mean and variance, to forecast the VaR and ES. We use GARCH dynamics for the conditional mean and variance to build our models, using the standardized residual. The copula forecasting model is:

$$Y_t = \mu_t + \sigma_t \eta_t \tag{27}$$

$$\sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \gamma \eta_{t-1}^2 \tag{28}$$

$$\eta_t \sim iid\, F_{\eta}(0,1) \tag{29}$$

We use the NGARCH model discussed above to consider leverage effects in forecasting:

$$\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \gamma\sigma_{t-1}^2(\eta_{t-1} - \theta)^2 \tag{30}$$

Y_t denotes the portfolios' return, where μ_t is specified to the ARMA model and σ_t^2 is specified to the GARCH model. $F_\eta(0,1)$ denotes the distribution of η_t. Given F_η, the forecasting of VaR and ES can be estimated as:

$$v_t = \mu_t + a\sigma_t, \; where\, a = F_\eta^{-1}(\alpha) \tag{31}$$

$$e_t = \mu_t + b\sigma_t, \; where\, b = \mathbb{E}\left[\eta_t \mid \eta_t \leq a\right] \tag{32}$$

We consider three choices for F_η to describe the distributions of η_t:

$$\eta_t \sim iid\, Normal\, copula \tag{33}$$

$$\eta_t \sim iid\, Student\, t\, copula \tag{34}$$

$$\eta_t \sim iid\, Skewed\, t\, copula \tag{35}$$

To estimate the parameters (a, b), we use the Monte Carlo simulation. We use simulation to define the quantile function and employ 1,000,000 replications using the equation below. Thereafter, we sort the replications to obtain the quantile value a and apply equation (31).

$$X = Z \tag{36}$$

where Z is the multi-normal distribution simulated by the correlation matrix from the normal copula.

$$X = \sqrt{W}Z \tag{37}$$

where Wfollows an inverse Gamma $IG(v/2, v/2)$ distribution, and v is from the student t copula model.

$$X = \sqrt{W}Z + \lambda W \tag{38}$$

where λ denotes a $N \times 1$ asymmetry parameter vector, and (v, λ) are all from the skewed t copula model.

Here μ_t is the mean value of $Y_t$and the $\sigma_t$is estimated by using the GARCH or NAGARCH model. Hence, we obtain the new forecasting values using the copula model. As for the univariate models, we apply the same GARCH model to esitmate the paramters. The main different between univariate model and copula model is in the estimation of parameters (a, b).

The NAGARCH models work very well. These models can help to accommodate extreme changes. The factors MOM and VAL carry the highest risk, this is consistent with what we observe in Figure 1.

Table 10 shows the fit of copula models. The first section presents the average loss using the FZ-loss function following Fissler (2017). The loss function is shown below:

$$\begin{aligned} L_{FZ}(Y, v, e, G_1, G_2) &= \underset{(v,e)}{argmin} \left(1\left\{Y \leq v\right\} - \alpha\right) \left(G_1\left(v\right) - G_1\left(Y\right) + \frac{1}{\alpha} G_2\left(e\right) v\right) \\ & -G_2\left(e\right) \left(\frac{1}{\alpha} 1\left\{Y \leq v\right\} Y - e\right) - \mathcal{G}_2\left(e\right) \end{aligned} \tag{39}$$

where G_1 denotes the weakly increasing and G_2 denotes the strictly increasing and strictly positive. G_2 is the differential coefficient function of $\mathcal{G}_2$, $\mathcal{G}_2' = G_2$. Parameters v, e denote the VaR and ES.

Smaller average losses indicate a better fit of VaR and ES. The models that consider asymmetry, generally, have the lowest average loss. The NAGARCH copula models carry always the lowest average loss. This result may also suggest that, amongst the forex factors, leverage is important. In sum, skewness and tail dependence do affect forex portfolio risk. Managing the risk of a forex portfolio is complex and the models assuming normality may not fully capture risk in the presence of shocks. Asymmetry amongst forex factors is an important element and can help managing portfolio risk. Figure 9 shows the forecasted value at risk of three different skewed t models. The results suggests the VaR of NAGARCH skewed t model can catch the extreme change of the risk.

7 Conclusion

We run a large and detailed study on the dependence structure amongst some of the most widely used forex factors. These factors are also very relevant to the hedge fund industry when designing forex trading strategies. We show that the dependence structure amongst the forex factors is more complex than what has been assumed in the literature. Asymmetry and time dependence are economically relevant. To evaluate the economic cost to a hedge fund of ignoring these features, we have considered two examples: forex portfolio management and forex portfolio risk management and show that adding asymmetry and time-varying dependence amongst the factors improves portfolio performance and risk management. Our results contribute to both the academic literature, for example cross sectional fx asset pricing and fx portfolio analysis as we shed some new light on the dependence structure across widely used forex factor, and it is also relevant to hedge funds when designing forex trading strategies.

References

Ang, Andrew and Geert Bekaert (2002). “International asset allocation with regime shifts”. In: *The Review of Financial Studies* 15.4, pp. 1137–1187.

Ang, Andrew and Joseph Chen (2002). “Asymmetric correlations of equity portfolios”. In: *Journal of financial Economics* 63.3, pp. 443–494.

Barroso, Pedro and Pedro Santa-Clara (2015a). “Beyond the carry trade: Optimal currency portfolios”. In: *Journal of Financial and Quantitative Analysis* 50.5, pp. 1037–1056.

– (2015b). “Momentum has its moments”. In: *Journal of Financial Economics* 116.1, pp. 111–120.

Cerrato, Mario et al. (2017a). “Relation between higher order comoments and dependence structure of equity portfolio”. In: *Journal of Empirical Finance* 40, pp. 101–120.

– (2017b). “The joint credit risk of UK global-systemically important banks”. In: *Journal of Futures Markets* 37.10, pp. 964–988.

Chen, Xiaohong and Yanqin Fan (2006). “Estimation and model selection of semiparametric copula-based multivariate dynamic models under copula misspecification”. In: *Journal of econometrics* 135.1-2, pp. 125–154.

Christoffersen, Peter, Vihang Errunza, et al. (2012). “Is the potential for international diversification disappearing? A dynamic copula approach”. In: *The Review of Financial Studies* 25.12, pp. 3711–3751.

Christoffersen, Peter and Hugues Langlois (2013). “The joint dynamics of equity market factors”. In: *Journal of Financial and Quantitative Analysis* 48.5, pp. 1371–1404.

Daniel, Kent and Tobias J Moskowitz (2016). “Momentum crashes”. In: *Journal of Financial Economics* 122.2, pp. 221–247.

Demarta, Stefano and Alexander J McNeil (2005). “The t copula and related copulas”. In: *International statistical review* 73.1, pp. 111–129.

Engle, Robert (2002). “Dynamic conditional correlation: A simple class of multivariate generalized autoregressive conditional heteroskedasticity models”. In: *Journal of Business & Economic Statistics* 20.3, pp. 339–350.

Engle, Robert F and Victor K Ng (1993). “Measuring and testing the impact of news on volatility”. In: *The journal of finance* 48.5, pp. 1749–1778.

Engle, Robert, Neil Shephard, and Kevin Sheppard (2009). “Fitting vast dimensional time-varying covariance models”. In:

Fama, Eugene F and Kenneth R French (1993). “Common risk factors in the returns on stocks and bonds”. In: *Journal of financial economics* 33.1, pp. 3–56.

Fissler, Tobias (2017). “On higher order elicitability and some limit theorems on the Poisson and Wiener space”. PhD thesis.

Hansen, Bruce E (1994). “Autoregressive conditional density estimation”. In: *International Economic Review*, pp. 705–730.

Jordà, Òscar and Alan M Taylor (2012). “The carry trade and fundamentals: Nothing to fear but FEER itself”. In: *Journal of International Economics* 88.1, pp. 74–90.

Kroencke, Tim A, Felix Schindler, and Andreas Schrimpf (2014). “International diversification benefits with foreign exchange investment styles”. In: *Review of Finance* 18.5, pp. 1847–1883.

Longin, Francois and Bruno Solnik (2001). “Extreme correlation of international equity markets”. In: *The journal of finance* 56.2, pp. 649–676.

Lustig, Hanno, Nikolai Roussanov, and Adrien Verdelhan (2011). “Common risk factors in currency markets”. In: *The Review of Financial Studies* 24.11, pp. 3731–3777.

– (2014). “Countercyclical currency risk premia”. In: *Journal of Financial Economics* 111.3, pp. 527–553.

Menkhoff, Lukas et al. (2012). “Currency momentum strategies”. In: *Journal of Financial Economics* 106.3, pp. 660–684.

Patton, Andrew J (2004). “On the out-of-sample importance of skewness and asymmetric dependence for asset allocation”. In: *Journal of Financial Econometrics* 2.1, pp. 130–168.

– (2006). “Modelling asymmetric exchange rate dependence”. In: *International economic review* 47.2, pp. 527–556.

Patton, Andrew J, Johanna F Ziegel, and Rui Chen (2019). “Dynamic semiparametric models for expected shortfall (and value-at-risk)”. In: *Journal of Econometrics* 211.2, pp. 388–413.

Scotti, Chiara and Sigridur Benediktsdottir (2009). “Exchange rates dependence: what drives it?” In: *FRB International Finance Discussion Paper* 969.

Table 1 – Description Statistics of Weekly Factor Return

We report the mean, volatility, skewness, kurtosis and autocorrelation and cross-correlation for logged weekly return of four factors. The period of the sample is from January 1, 1989, to March 20, 2020. The significant correlation is marked by * and ** denoting the 5% and 1% levels.

Sample Moments	DOL	HML	MOM	VAL
Annualized mean	-0.0033	0.1817	0.0790	0.0336
Weekly mean	-0.0001	0.0035	0.0015	0.0006
Annualized volatility	0.0646	0.0826	0.0935	0.0964
Weekly volatility	0.0090	0.0114	0.0130	0.0134
skewness	-0.3476	-0.3983	-0.1451	0.2589
Kurtosis	4.7311	5.1602	6.9311	6.8570
Autocorrelation				
First-order	0.0410	-0.0055	-0.0101	0.0923**
Second-order	0.0354	0.0922**	0.0640*	0.1751**
Third-order	0.0228	0.0781**	0.0870**	0.1436**
Cross Correlations				
DOL	1.0000	0.3120**	-0.0786*	-0.1596**
HML	0.3120**	1.0000	0.0522*	-0.2655**
MOM	-0.0786*	0.0522*	1.0000	0.5585**
VAL	-0.1596**	-0.2655**	0.5585**	1.0000

Table 2 – different group and period of four factors' correlations

We present the different group and period correlations to understand the reason for the positive correlation between HML and MOM or negative correlation between HML and VAL. The first section presents the correlation of the group of developed country factors, while the second section show the correlations from developing country factors. The last section is the cross-section data without the 2008 financial crrisis.

	developed countries				developing countries				without financial		
	DOL	HML	MOM	VAL	DOL	HML	MOM	VAL	DOL	HML	M
DOL	1.0000	0.2770	-0.1377	-0.0580	1.0000	0.4045	-0.0462	-0.0011	1.0000	0.2918	-0.
HML	0.2770	1.0000	-0.1582	0.0106	0.4045	1.0000	0.2248	0.3090	0.2918	1.0000	0.0
MOM	-0.1377	-0.1582	1.0000	0.1443	-0.0462	0.2248	1.0000	0.3187	-0.0604	0.0718	1.0
VAL	-0.0580	0.0106	0.1443	1.0000	-0.0011	0.3090	0.3187	1.0000	-0.1755	-0.3292	0.4

Table 3 – Estimation table of normal residuals

We report parameter estimates and model diagnostics for the AR-GARCH model with normal shocks. Standard errors which are in parentheses are calculated from the outer product of the gradient at the optimum parameter values. The model estimated is $r_{j,t} = \phi_{0,j} + \phi_{i,j} r_{j,t-1} + \sigma_{j,t}\epsilon_{j,t}$, where $\sigma^2_{j,t} = \omega_j + \beta_j \sigma^2_{j,t-1} + \alpha_j \sigma^2_{j,t-1}(\epsilon_{j,t-1} - \theta_j)^2$. Here ω is fixed by variance targeting, and variance persistence denotes the sum of parameters of the model.We also provide the p-value for Ljung-Box (L-B) tests of the residuals and absolute residuals by 20 lags. The empirical skewness and excess kurtosis of the residuals are compared to the model implied levels from the normal model.

Parameter Estimates	DOL	HML	MOM	VAL
ϕ_0	-0.0003	0.0031	0.0012	0.0005
	(0.0008)	(0.0003)	(0.0003)	(0.0003)
ϕ_1	0.0435	0.0152	0.0118	0.1075
	(0.0407)	(0.0306)	(0.0279)	(0.0315)
α	0.0000	0.0000	0.0000	0.0000
	(0.0000)	(0.0000)	(0.0000)	(0.0000)
β	0.9330	0.9519	0.9106	0.0022
	(0.5577)	(0.0189)	(0.0942)	(0.0023)
θ	-0.3675	0.0842	0.4922	0.1167
	(0.2199)	(0.1203)	(0.1116)	(0.0160)
κ	/	/	/	/
υ	/	/	/	/
Diagnostics				
Log-likelihood	5131.1000	4714.6000	4572.5000	4451.7000
Variance persistence	0.9330	0.9519	0.9106	0.0022
L-B(20) p-value	0.1366	0.0000	0.0000	0.0000
Absolute L-B(20) p-value	0.0000	0.0000	0.0000	0.0000
Empirical skewness	-0.3305	-0.3874	-0.1572	0.1523
Model skewness	0.0000	0.0000	0.0000	0.0000
Empirical excess kurtosis	4.7064	5.1739	7.0292	6.9138
Model excess kurtosis	0.0000	0.0000	0.0000	0.0000

Table 4 – Estimation table of skewed t residuals

We report parameter estimates and model diagnostics for the AR-GARCH model with skewed t shocks. Standard errors which are in parentheses are calculated from the outer product of the gradient at the optimum parameter values. The model estimated is $r_{j,t} = \phi_{0,j} + \phi_{i,j} r_{j,t-1} + \sigma_{j,t}\epsilon_{j,t}$, where $\sigma^2_{j,t} = \omega_j + \beta_j \sigma^2_{j,t-1} + \alpha_j \sigma^2_{j,t-1}(\epsilon_{j,t-1} - \theta_j)^2$. Here ω is fixed by variance targeting, and variance persistence denotes the sum of parameters of the model.We also provide the p-value for Ljung-Box (L-B) tests of the residuals and absolute residuals by 20 lags. The empirical skewness and excess kurtosis of the residuals are compared to the model implied levels from the asymmetric model.

Parameter Estimates	DOL	HML	MOM	VAL
ϕ_0	0.0086	0.0187	0.0127	0.0010
	(4.4267)	(39.8799)	(19.0416)	(0.0003
ϕ_1	0.1205	-0.9963	-0.3788	0.0842
	(21.1115)	(626.7629)	(95.5201)	(0.0290
α	0.1420	0.1426	0.1357	0.0000
	(99.8384)	(2238.7000)	(10.3699)	(0.0000
β	0.0487	0.0495	0.0872	0.9278
	(0.7801)	(1001.0000)	(16.2953)	(0.0310
θ	0.0980	0.1031	0.1537	0.0326
	(6.9545)	(340.2772)	(17.5421)	(0.1055
κ	0.6267	0.6386	0.7202	0.0294
	(984.0096)	(4693.0000)	(124.6940)	(0.0393
υ	8.5314	9.4683	6.6711	8.6676
	(313.4826)	(7898.4000)	(1234.2000)	(1.6343
Diagnostics				
Log-likelihood	3092.6000	2768.6000	2968.2000	4542.9000
Variance persistence	0.1907	0.1921	0.2229	0.9278
L-B(20) p-value	0.0188	0.0000	0.0000	0.0000
Absolute L-B(20) p-value	0.0000	0.0000	0.0000	0.0000
Empirical skewness	-0.2970	-0.1663	0.1166	0.1773
Model skewness	1.3041	1.2484	1.6589	0.0770
Empirical excess kurtosis	4.6803	4.1084	4.9599	6.8957
Model excess kurtosis	6.2522	5.7910	8.9601	4.2916

Table 5 – Estimation results for copula models with composite method

This table presents parameter estimates for the dependence models of the residuals from the NAGARCH model for the period January 1, 1989, to March 20, 2020. All models are estimated by maximum likelihood. Standard errors (in parentheses) are computed using the methodology of R. Engle, Shephard, and Sheppard (2009). The last line presents the pseudo-likelihood ratio test statistics. We followed X. Chen and Fan (2006) for the null hypothesis that the asymmetry parameters in skewed t copula are all equal to 0. The * and ** denote the significant levels of 5% and 1%.

4 factors						
	constant			dynamic		
	normal	t	skewed t	normal	t	skewed t
υ		9.1563	7.2539		6.0653	6.4937
		(2.2558)	(0.1277)		(0.9437)	(0.2823)
λ_{DOL}			-0.0009			-0.0025
			(0.0002)			(0.0013)
λ_{HML}			0.0015			0.0031
			(0.0005)			(0.0007)
λ_{MOM}			-0.0013			-0.0022
			(0.0162)			(0.0051)
λ_{VAL}			-0.0095			-0.0002
			(0.0026)			(0.0106)
β_c				0.8115	0.7951	0.8087
				(0.0338)	(0.0130)	(0.0212)
α_c				0.0247	0.0369	0.0304
				(0.0069)	(0.0016)	(0.0027)
ρ(DOL,HML)	0.1615	0.1489	0.1474	0.2058	0.2131	0.2129
ρ(DOL,MOM)	0.0184	0.0403	0.0403	0.0180	-0.0025	-0.0005
ρ(DOL,VAL)	-0.0531	-0.0050	-0.0036	-0.0537	-0.0990	-0.0947
ρ(HML,MOM)	0.0814	0.0948	0.0946	0.1058	0.0800	0.0820
ρ(HML,VAL)	-0.1598	-0.1605	-0.1584	-0.1822	-0.1904	-0.1896
ρ(MOM,VAL)	0.5533	0.5597	0.5575	0.5544	0.5683	0.5676
Model Properties						
Correlation persistence	0.0000	0.0000	0.0000	0.8362	0.8320	0.8391
Log-likelihood	334.9903	544.7528	551.9646	521.3643	602.8038	610.3774
No. of parameters	6.0000	7.0000	11.0000	8.0000	9.0000	13.0000
Pseudo-likelihood			15.2801**			12.4748**

Table 6 – Out of sample investment results

The period of the out-of-sample is from April 1, 1994, to March 20, 2020. For each level of relative risk aversion, the performance of the three copula models is compared to the benchmark normal distribution. Panels A,B and C show the results for a relative risk aversion coefficient of 3,7 and 10, respectively. We report the realized moments of the portfolio returns, the average turnover, as well as the certainty equivalent.

	dynamic correlation models				
	normal distribution	normal copula	student t copula	skewed t copula	equal weighted
Panel A. gamma=3					
Annualized mean(%)	18.6041	18.5902	18.6048	18.8868	7.3950
Annualized volatility(%)	0.0166	0.0166	0.0166	0.0162	0.0041
skewness	-0.1503	-0.1505	-0.1525	-0.0346	0.7174
Kurtosis	2.9245	2.9257	2.9320	2.5787	3.1424
Average turnover(%)	1.8146	1.8802	1.9300	1.7100	0.0000
CE(basis point)	35.6977	35.6709	35.6991	36.2452	14.2164
Annualized diff in CE(%)	–	–	–	0.2847	–
Panel B. gamma=7					
Annualized mean(%)	18.4487	18.4349	18.5316	18.6866	7.3950
Annualized volatility(%)	0.0167	0.0167	0.0167	0.0162	0.0041
skewness	-0.1285	-0.1299	-0.1482	0.0049	0.7174
Kurtosis	2.9503	2.9526	2.9547	2.6629	3.1424
Average turnover(%)	2.1904	2.2436	2.2159	2.1294	0.0000
CE(basis point)	35.2920	35.2649	35.4512	35.7602	14.2099
Annualized diff in CE(%)	–	–	–	0.2435	–
Panel C. gamma=10					
Annualized mean(%)	18.3096	18.3115	18.4622	18.6422	7.3950
Annualized volatility(%)	0.0168	0.0168	0.0168	0.0168	0.0041
skewness	-0.1264	-0.1283	-0.1571	-0.1413	0.7174
Kurtosis	2.9185	2.9218	2.9117	2.8456	3.1424
Average turnover(%)	2.2007	2.1922	2.2228	2.3522	0.0000
CE(basis point)	34.9407	34.9443	35.2332	35.5814	14.2051
Annualized diff in CE(%)	–	–	–	0.3332	–

Table 7 – Out of sample investment with transaction cost

The period of the out-of-sample is from April 1, 1994, to March 20, 2020. For each level of relative risk aversion, the performance of the three copula models is compared to the benchmark normal distribution. Panels A,B and C show the results for a relative risk aversion coefficient of 3,7 and 10, respectively. We report the realized moments of the portfolio returns with cost, the average turnover, as well as the certainty equivalent.

	dynamic correlation models with cost				
	normal distribution	normal copula	student t copula	skewed t copula	equal weighted
Panel A. gamma=3					
Annualized mean(%)	17.7378	17.7003	17.6939	18.0688	7.3950
Annualized volatility(%)	1.7386	1.7424	1.7455	1.6853	0.0041
skewness	-0.2020	-0.2010	-0.2123	-0.0574	0.7174
Kurtosis	2.9719	2.9645	2.9875	2.6052	3.1424
Average turnover(%)	1.8146	1.8802	1.9300	1.7100	0.0000
CE(basis point)	34.0243	33.9519	33.9393	34.6662	14.2164
Annualized diff in CE(%)	–	–	–	0.3338	–
Panel B. gamma=7					
Annualized mean(%)	17.3506	17.3091	17.3994	17.5951	7.3950
Annualized volatility(%)	1.7577	1.7624	1.7535	1.6928	0.0041
skewness	-0.1675	-0.1728	-0.1680	-0.0093	0.7174
Kurtosis	2.9482	2.9569	2.9267	2.7304	3.1424
Average turnover(%)	2.1904	2.2436	2.2159	2.1294	0.0000
CE(basis point)	33.1593	33.0783	33.2541	33.6448	14.2099
Annualized diff in CE(%)	–	–	–	0.2525	–
Panel C. gamma=10					
Annualized mean(%)	17.1896	17.1886	17.3188	17.4373	7.3950
Annualized volatility(%)	1.7700	1.7696	1.7668	1.7629	0.0041
skewness	-0.1657	-0.1645	-0.1719	-0.1557	0.7174
Kurtosis	2.9228	2.9147	2.8781	2.8346	3.1424
Average turnover(%)	2.2007	2.1922	2.2228	2.3522	0.0000
CE(basis point)	32.7564	32.7547	33.0061	33.2354	14.2051
Annualized diff in CE(%)	–	–	–	0.2491	–

Table 8 – Out of sample investment in developed currencies

The period of the out-of-sample is from April 1, 1994, to March 20, 2020.. For each level of relative risk aversion, the performance of the three copula models is compared to the benchmark normal distribution. Panels A,B and C show the results for a relative risk aversion coefficient of 3,7 and 10, respectively. We report the realized moments of the portfolio returns, the average turnover, as well as the certainty equivalent.

	developed countries models				
	normal distribution	normal copula	student t copula	skewed t copula	equal weighted
Panel A. gamma=3					
Annualized mean(%)	16.5457	16.5447	16.5022	16.6840	4.0633
Annualized volatility(%)	0.0182	0.0182	0.0183	0.0182	0.0043
skewness	-0.4936	-0.4941	-0.4906	-0.5003	0.3027
Kurtosis	2.8998	2.9021	2.8891	2.9011	2.2994
Average turnover(%)	2.7972	2.8082	2.6592	2.5465	0.0000
CE(basis point)	31.7229	31.7210	31.6389	31.9890	7.8087
Annualized diff in CE(%)	–	–	–	0.1384	–
Panel B. gamma=7					
Annualized mean(%)	16.2548	16.2608	16.1788	16.2690	4.0633
Annualized volatility(%)	0.0182	0.0182	0.0184	0.0182	0.0043
skewness	-0.4429	-0.4457	-0.4464	-0.4540	0.3027
Kurtosis	2.8775	2.8837	2.8400	2.8702	2.2994
Average turnover(%)	2.8179	2.7921	2.8059	2.3974	0.0000
CE(basis point)	31.0357	31.0474	30.8865	31.0629	7.8016
Annualized diff in CE(%)	–	–	–	0.0141	–
Panel C. gamma=10					
Annualized mean(%)	16.1396	16.1490	16.0834	16.1621	4.0633
Annualized volatility(%)	0.0183	0.0183	0.0184	0.0184	0.0043
skewness	-0.4256	-0.4254	-0.4229	-0.4226	0.3027
Kurtosis	2.8316	2.8289	2.7956	2.7986	2.2994
Average turnover(%)	2.7410	2.7062	2.7327	2.4695	0.0000
CE(basis point)	30.7151	30.7327	30.6034	30.7549	7.7962
Annualized diff in CE(%)	–	–	–	0.0207	–

Table 9 – Out of sample investment in developing currencies

The period of the out-of-sample is from April 1, 1994, to March 20, 2020. For each level of relative risk aversion, the performance of the three copula models is compared to the benchmark normal distribution. Panels A,B and C show the results for a relative risk aversion coefficient of 3,7 and 10, respectively. We report the realized moments of the portfolio returns, the average turnover, as well as the certainty equivalent.

	developing countries models				
	normal distribution	normal copula	student t copula	skewed t copula	equal weighted
Panel A. gamma=3					
Annualized mean(%)	23.8652	23.8711	23.8622	23.9173	10.1632
Annualized volatility(%)	0.0228	0.0228	0.0228	0.0228	0.0074
skewness	-0.0466	-0.0445	-0.0464	-0.0475	1.0550
Kurtosis	2.7375	2.7381	2.7372	2.7331	3.4387
Average turnover(%)	1.9524	1.9990	2.0320	1.8765	0.0000
CE(basis point)	45.3976	45.7572	45.7398	45.8461	19.5289
Annualized diff in CE(%)	–	–	–	0.2332	–
Panel B. gamma=7					
Annualized mean(%)	23.7773	23.7775	23.7976	23.8410	10.1632
Annualized volatility(%)	0.0232	0.0232	0.0232	0.0231	0.0074
skewness	-0.1144	-0.1141	-0.1158	-0.1137	1.0550
Kurtosis	2.7709	2.7700	2.7811	2.7852	3.4387
Average turnover(%)	2.3941	2.3756	2.4624	2.3812	0.0000
CE(basis point)	45.3650	45.3654	45.4049	45.4898	19.5078
Annualized diff in CE(%)	–	–	–	0.0275	–
Panel C. gamma=10					
Annualized mean(%)	23.9086	23.9090	23.9154	23.9951	10.1632
Annualized volatility(%)	0.0230	0.0231	0.0230	0.0230	0.0074
skewness	-0.1484	-0.1508	-0.1537	-0.1571	1.0550
Kurtosis	2.9374	2.9403	2.9567	2.9740	3.4387
Average turnover(%)	2.5953	2.6082	2.6819	2.6416	0.0000
CE(basis point)	45.4693	45.4698	45.4829	45.6374	19.4921
Annualized diff in CE(%)	–	–	–	0.0874	–

Table 10 – Average loss of copula forecasting models

The left-hand panel of this table presents the average losses, using the FZ loss function following Fissler (2017), for four factors return series, over the out-of-sample period from January 7 1994 to January 5 2018, for nine different forecasting models. The first three rows correspond to the univariate distribution forecast, the next three rows correspond to dynamic copula forecasts, the last three rows give the forecasting results from NAGARCH dynamic copula models.

	Avg loss			
	DOL	HML	MOM	VAL
GCH-n-simple	-3.9370	-3.7620	-3.4500	-3.6870
GCH-skt-simple	-3.9260	-3.7510	-3.4410	-3.6890
GCH-emp-simple	-3.9330	-3.7410	-3.4350	-3.6860
GCH-n-dcc	-3.9390	-3.7630	-3.4450	-3.6870
GCH-t-dcc	-3.7860	-3.7970	-3.6070	-3.5500
GCH-skt-dcc	-3.7320	-3.7660	-3.5880	-3.5760
NGCH-n-dcc	-2.5760	-3.6840	-2.1880	-3.1650
NGCH-t-dcc	-2.6910	-3.7020	-2.3540	-3.1960
NGCH-skt-dcc	-2.6810	-3.6960	-2.1530	-3.2000

Figure 1 – Time series plot for 4 factors

The figure below illustrates the time series of weekly returns of each factor for the period January 1, 1989, to March 20, 2020.

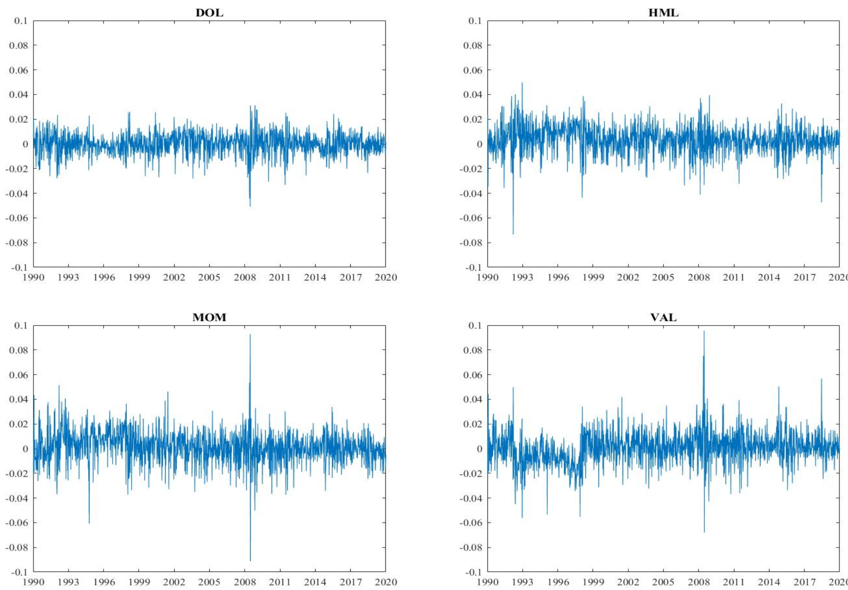

Figure 2 – Quantile-Quantile Plots for 4 factors

For each observation we scatter plot the empirical quantile on the vertical axis against the corresponding quantile from the standard normal distribution on the horizontal axis. If returns are normally distributed, then the data points will fall randomly around the 45° line ,which is marked by dashes.

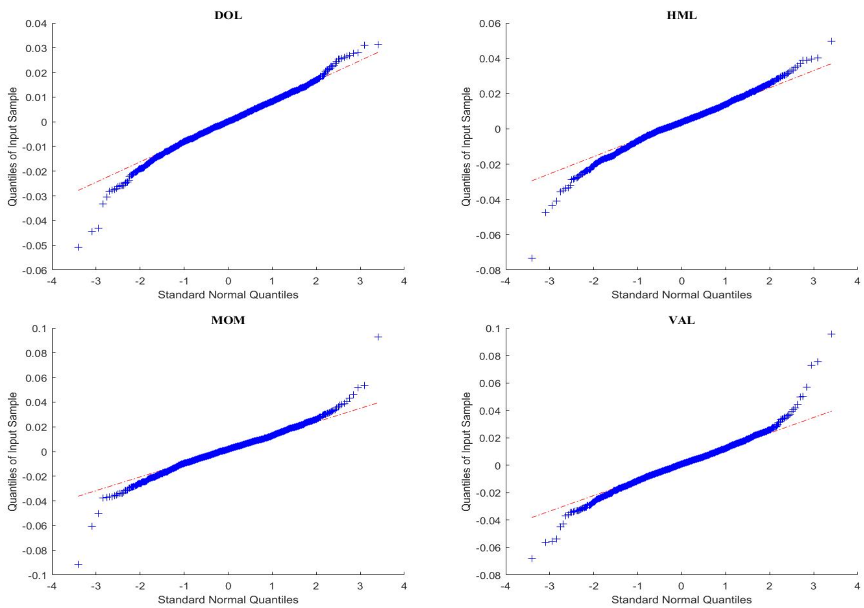

Figure 3 – Threshold correlation for 4 factors

Figure 3 presents threshold correlations between the 4 factors . Our sample consists of weekly returns from January 1, 1989, to March 20, 2020. The continuous line represents the correlations when both variables are below (above) a threshold when this threshold is below (above) the median. The dashed line represents the threshold function for a bivariate normal distribution using the linear correlation coefficient from the data.

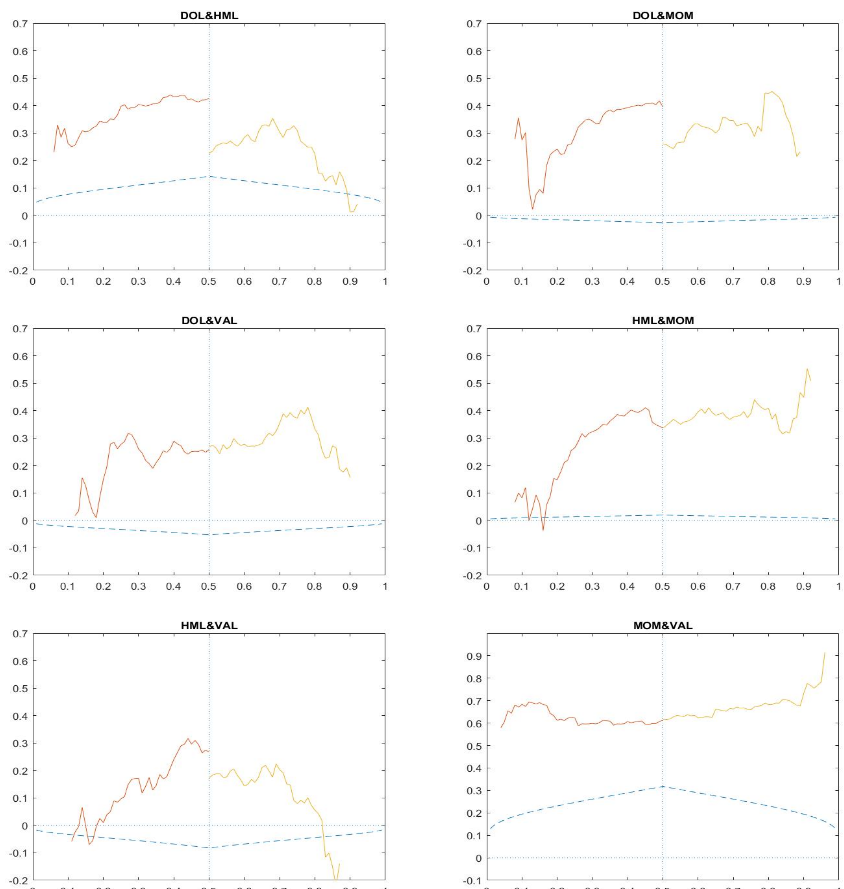

Figure 4 – Autocorrelation 4 factors and the absolute value of 4 factors

Autocorrelation of weekly returns (dashed line) and absolute returns (solid line) from January 1, 1989, to March 20, 2020. The horizontal dotted lines provide a 95 confidence interval around 0.

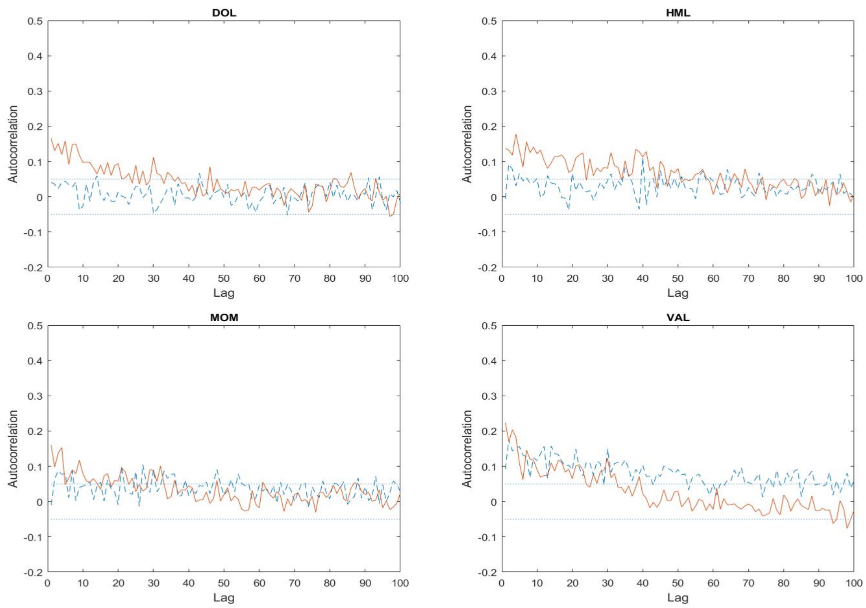

Figure 5 – Autocorrelation graph of residual series

Autocorrelation of AR-FARCH residuals (dashed line) and absolute residuals (solid line) from January 1, 1989, to March 20, 2020. The horizontal dotted lines provide a 95 confidence interval around 0.

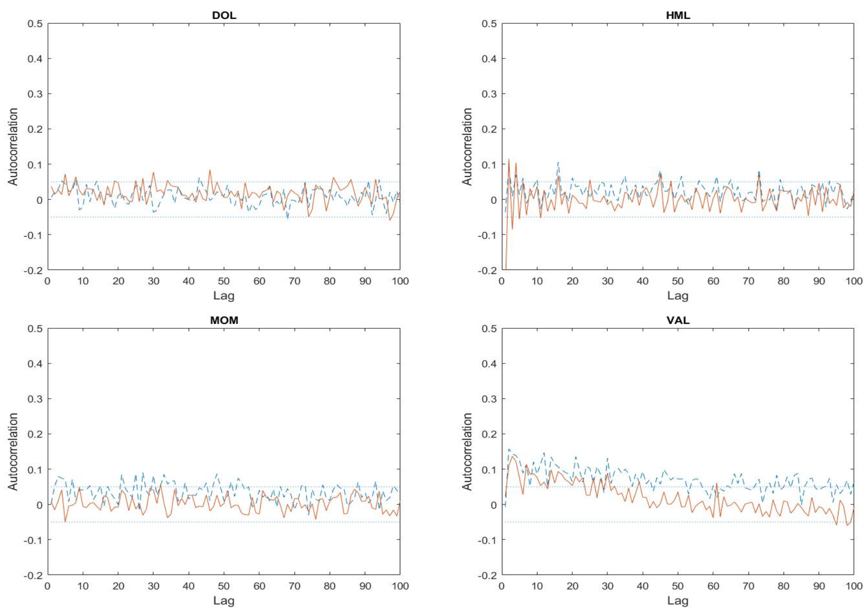

Figure 6 – QQ plot of residuals series

For each observation we scatter plot the empirical quantile on the vertical axis against the corresponding quantile from the skewed t distribution on the horizontal axis. If the AR-GARCH residuals adhere to the skewed t distribution, then the data points will fall on the 45^{o} line, which is marked by dashes. The parameters for the skewed t distribution are from Table 3.

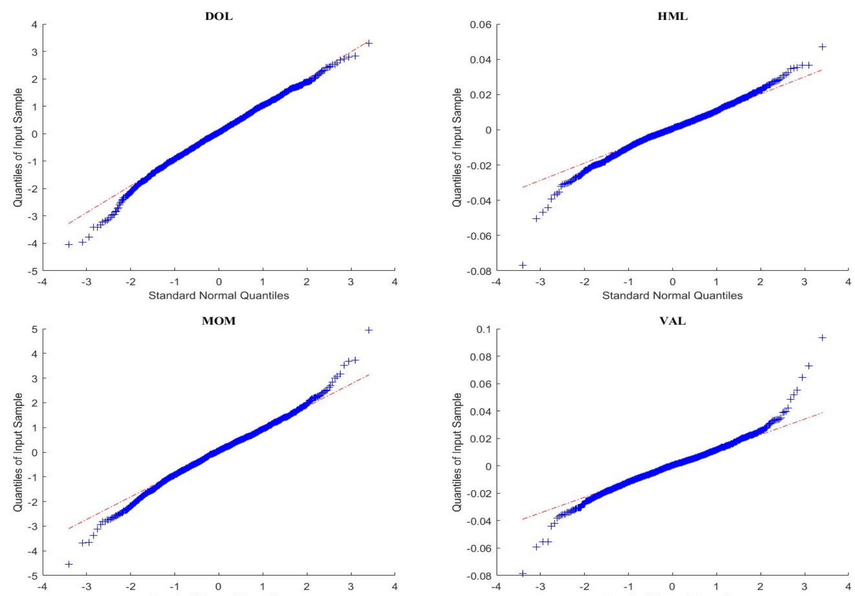

Figure 7 – Skewed t copula dynamic correlations with composite method

We report dynamic conditional copula correlation for each pair of factors from January 1, 1989, to March 20, 2020. The correlations are obtained by estimating the dynamic skewed t copula model on the factor return residuals from the AR-GARCH model. This sample is used in estimation of the models.

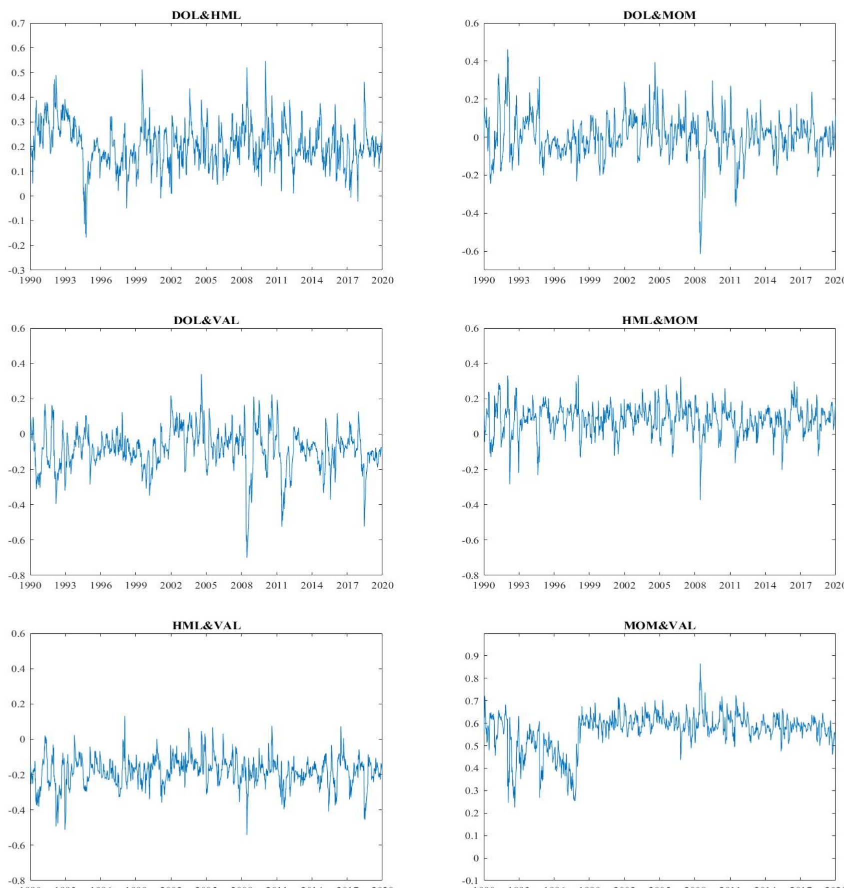

Figure 8 – Threshold Correlations for Factor Residuals and Copula Models

We present threshold correlations computed on AR-GARCH residuals from January 1, 1989, to March 20, 2020. The thick continuous line represents the empirical correlation. The threshold correlation functions are computed for thresholds for which there are at least 24 data points available. We compared the empirical correlations to those implied by the normal copula and the constant t and skewed t copulas.

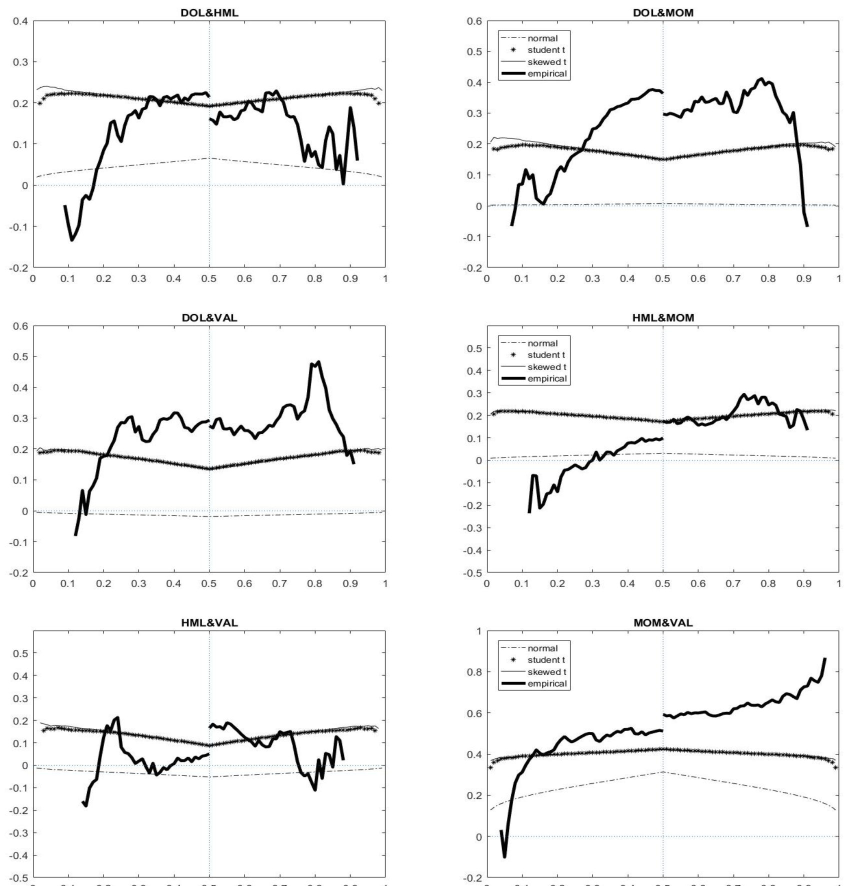

Figure 9 – VaR of factors

We report the forecasted value at risk of four factors of three different forecasting model, simple skewed t, multi skewed t and NAGARCH skewed t model. The blue line give the VaR of simple skewed t model. The red line show the results of multi skewed t model. The yellow line denotes the VaR of NAGARCH skewed t forecasting model.